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M2 to D2

Sunil Mukhi and Constantinos Papageorgakis

Tata Institute of Fundamental Research,
Homi Bhabha Rd, Mumbai 400, 005, India
E-mail: mukhi@tifr.res.in, costis@theory.tifr.res.in

Abstract: We examine the recently proposed "3-algebra" field theory for multiple M2-branes and show that when a scalar field valued in the 3-algebra develops a vacuum expectation value, the resulting Higgs mechanism has the novel effect of promoting topological (Chern-Simons) to dynamical (Yang-Mills) gauge fields. This leads to a precise derivation of the maximally supersymmetric Yang-Mills theory on multiple D2-branes and thereby provides a relationship between 3-algebras and Yang-Mills theories. We discuss the physical interpretation of this result.

Keywords: M-Theory, D-branes.

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Contents

1. Introduction

The world-volume theory on multiple M2-branes has remained mysterious since the inception of M-theory over a decade ago. It is expected to be the conformal-invariant IR fixed point of the D2-brane world-volume theory, which to lowest order is a maximally supersymmetric Yang-Mills theory in $2+1$ dimensions. The M2 theory should have 8 transverse scalar fields as its bosonic content, while the D2 theory is known to have 7 scalar fields and a gauge field (for a pedagogical review of M-theory see [1], for a review of M-branes see ref. [2]).

For the Abelian case, both the D2 theory and the M2 theory are free and in this case the relation between them follows by performing an Abelian duality on the gauge field of the D2 brane, which converts it into the 8th scalar on the M2-brane [3 – 6]. The analogue of this relation has not been found for the non-Abelian case so far, given the absence of a known interacting CFT for multiple M2-branes.

Recently a concrete proposal has been made for the world-volume theory on multiple M2 branes [7, 8] following preliminary ideas in refs. [9, 10]. In this proposal the field content is a collection of scalars, fermions and gauge fields transforming under a "3-algebra", a generalisation of a Lie algebra with a triple bracket replacing the commutator and a 4-index structure constant replacing the usual 3-index structure constant of a Lie algebra. There is also a bilinear "fundamental identity" replacing the Jacobi identity of a Lie algebra.[1] The scalars and fermions are dynamical and coupled via a sextic self-coupling and a 2-scalar-2-fermion analogue of a Yukawa coupling. The gauge field, in contrast, is topological and has a Chern-Simons self-coupling as well as minimal couplings to the matter fields. It contributes no on-shell degrees of freedom.

The proposed action is maximally supersymmetric (the supersymmetry algebra closes on-shell) and classically conformal invariant. It has no free parameters and the structure

[1]In our work we use the notation, conventions and terminology of ref. [7].

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constants of the 3-algebra are quantised [11], strongly suggesting that conformal invariance is exact at the quantum level. The theory has an elegant and unique structure which makes it a very compelling candidate to be the multiple M2-brane theory. In addition it has some features which might not have been anticipated on general grounds, for example the gauge symmetry associated to the Chern-Simons gauge field.

Nevertheless the proposal is incomplete for a few reasons. Only a single 3-algebra (called $\mathcal{A}_4$) is explicitly known, and the vacuum moduli space of the postulated theory has two free parameters. If we add a zero mode supermultiplet (having vanishing 3-algebra bracket with all other fields) following the usual procedure in D-brane theories, we get altogether three parameters, which has been interpreted in ref. [11] as corresponding to *three* M2-branes. A possible interpretation suggested there was that there is no interacting theory for two M2-branes, and therefore the IR limit of maximal SYM in $2+1$ dimensions is trivial, a rather dramatic hypothesis for which no evidence is known.[2] Another, apparently independent, limitation of the proposed M2-brane theory is that despite some attempts [8, 12], it has not been possible to recover the multiple-D2-brane theory from it after compactifying a transverse direction.

In this work we make an observation that relates the proposed multiple M2-brane theory to the strongly coupled maximally supersymmetric Yang-Mills theory in 2+1 dimensions. This happens when a scalar field *develops a VEV in a 3-algebra direction*. The resultant Higgsing leads to an SU(2) D2-brane theory including a *dynamical* SU(2) gauge field, plus a decoupled Abelian degree of freedom. Pleasingly, the SU(2) gauge field is a part of the Chern-Simons gauge field of the original 3-algebra theory, which becomes dynamical via a generalised Higgs mechanism as we will demonstrate. All interactions of the Yang-Mills theory, and no others, are found in the strong coupling limit. Finally, with minor changes the proposal extends to other 3-algebras, whose properties we can characterise but of which explicit examples are not yet known.

In what follows we review the multiple M2-brane action based on 3-algebras, then describe our results in some detail for the 3-algebra $\mathcal{A}_4$. Next we discuss the generalisation to arbitrary 3-algebras and conclude with some open questions.

2. The 3-algebra field theory

The maximally supersymmetric 3-algebra field theory in 2+1 dimensions [7, 8, 11] describes a set of bosonic fields $X^{A(I)}, A_\mu{}^{AB}$ and fermionic fields Ψ^A having (suppressed) spinor indices with respect to $SO(2,1)$ as well as $SO(8)$. Here the indices $\{A, B, \ldots\}$ take the values $1, \ldots, \dim_{\mathcal{A}}$ with $\dim_{\mathcal{A}}$ being the dimension of a 3-algebra, which we will leave unspecified for the moment, while $\{I, J, \ldots\} = 1, 2, \ldots, 8$ label the scalar fields corresponding to the 8 directions transverse to the M2-branes.

To write the action we first introduce 4-index structure constants f^{ABCD} associated with a formal, totally antisymmetric three-bracket over the three-algebra generators:

$$[T^A, T^B, T^C] = f^{ABC}{}_D T^D \tag{2.1}$$

[2] We are grateful to Shiraz Minwalla for stressing this point.

and a generalisation of the trace, "Tr" taken over the three-algebra indices, which provides an appropriate '3-algebra metric':

$$h^{AB} = \mathrm{Tr}(T^A, T^B)\ . \tag{2.2}$$

Then the 4-index structure constants satisfy the 'fundamental identity':

$$f^{AEF}{}_G f^{BCDG} - f^{BEF}{}_G f^{ACDG} + f^{CEF}{}_G f^{ABDG} - f^{DEF}{}_G f^{ABCG} = 0 \tag{2.3}$$

and are also completely antisymmetric under the exchange of indices:

$$f^{ABCD} = f^{[ABCD]}\ . \tag{2.4}$$

All information about the 3-algebra is contained in the structure constants, so we will write actions and equations of motion without referring again to the 3-bracket. This avoids the question of what algebraic structure (analogous to matrices) is encoded in the 3-bracket, and removes much of the mystery from the action — which is ultimately a set of couplings among multiplets of ordinary fields. This action is:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}D_\mu X^{A(I)} D^\mu X_A^{(I)} + \frac{i}{2}\overline{\Psi}^A \Gamma^\mu D_\mu \Psi_A + \frac{i}{4} f_{ABCD}\overline{\Psi}^B \Gamma^{IJ} X^{C(I)} X^{D(J)} \Psi^A \\ &-\frac{1}{12}\left(f_{ABCD} X^{A(I)} X^{B(J)} X^{C(K)}\right)\left(f_{EFG}{}^D X^{E(I)} X^{F(J)} X^{G(K)}\right) \\ &+\frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f_{ABCD} A_\mu{}^{AB} \partial_\nu A_\lambda{}^{CD} + \frac{2}{3} f_{AEF}{}^G f_{BCDG} A_\mu{}^{AB} A_\nu{}^{CD} A_\lambda{}^{EF}\right)\end{aligned} \tag{2.5}$$

where:

$$D_\mu X^{A(I)} = \partial_\mu X^{A(I)} + f^A{}_{BCD} A_\mu^{CD} X^{B(I)}\ . \tag{2.6}$$

It is invariant under the gauge transformations:

$$\begin{aligned}\delta X^{A(I)} &= -f^A{}_{BCD}\Lambda^{BC} X^{D(I)} \\ \delta\Psi^A &= -f^A{}_{BCD}\Lambda^{BC}\Psi^D \\ \delta(f_{AB}{}^{CD} A_\mu^{AB}) &= f_{AB}{}^{CD} D_\mu \Lambda^{AB}\end{aligned} \tag{2.7}$$

and the supersymmetries:

$$\begin{aligned}\delta X^{A(I)} &= i\,\overline{\epsilon}\,\Gamma^I \Psi^A \\ \delta\Psi^A &= D_\mu X^{A(I)}\Gamma^\mu\Gamma^I\epsilon + \frac{1}{6} f^A{}_{BCD} X^{B(I)} X^{C(J)} X^{D(K)} \Gamma^{IJK}\epsilon \\ \delta(f_{AB}{}^{CD} A_\mu^{AB}) &= i f_{AB}{}^{CD} X^{A(I)}\,\overline{\epsilon}\,\Gamma_\mu \Gamma_I \Psi^B\end{aligned} \tag{2.8}$$

where $\Gamma_{012}\epsilon = \epsilon$ and $\Gamma_{012}\Psi^A = -\Psi^A$.

A potentially puzzling feature of this theory is that while the fundamental gauge field is A_μ^{AB}, it is the combination $\tilde{A}_\mu^{CD} = f_{AB}{}^{CD} A_\mu^{AB}$ that appears in the symmetry transformations and covariant derivatives, despite the fact that the Chern-Simons action cannot be written in terms of $\tilde{A}$ alone. This was explained in ref. [7] by noting that those

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variations in A that do not affect $\tilde{A}$ leave the Chern-Simons action invariant. Therefore in a subtle way, the theory depends only on the gauge field $\tilde{A}$.

The above theory is manifestly conformally invariant at the classical level. If the proposal that it describes M2-branes is correct then it must also be quantum mechanically conformal invariant. This is very plausible, though it has not yet been explicitly demonstrated.

The one 3-algebra that can be easily constructed (in fact, the only one constructed so far) has structure constants given by the 4-index totally antisymmetric symbol $f^{ABCD} = \varepsilon^{ABCD}$ with $A, B, C, D \in \{1, 2, 3, 4\}$. This is the lowest dimensional 3-algebra that one can write down and has been denoted $\mathcal{A}_4$. The action in this case has an SO(4) rotation invariance. For this action it was observed in ref. [11] that the vacuum moduli space, defined as the space of solutions to the equations:

$$f_{ABCD} X^{A(I)} X^{B(J)} X^{C(K)} = 0 \tag{2.9}$$

is given by:

$$X^{A(I)} = a^{(I)} \alpha^A + b^{(I)} \beta^A \tag{2.10}$$

where α^A, β^A are arbitrary elements of the $\mathcal{A}_4$ algebra and a^I, b^I are constant vectors. It was postulated that 3-algebras for M2-branes should be supplemented by a new "central" direction "0" such that $f^{0ABC} = 0$ for all A, B, C, and that the fields $X^{0(I)}, \Psi^0$ describe the overall centre-of-mass or zero mode of the M2-brane system. Adding in the zero mode for the special case of the $\mathcal{A}_4$ algebra, one finds a 3-parameter vacuum moduli space that was interpreted in ref. [11] as describing *three* M2-branes. We will re-examine this interpretation in the concluding section.

3. M2 to D2 for U(2)

In this section we re-examine the field theory based on the $\mathcal{A}_4$ 3-algebra and will find that it quite naturally describes a pair of 2-branes coupled via a supersymmetric Yang-Mills action, along with a free Abelian theory. The emergence of dynamical Yang-Mills interactions constitutes a sensitive check of the proposed M2-brane action and tests many of its detailed features, including its somewhat baroque Chern-Simons structure.

We start by assuming a scalar field in the 3-algebra theory develops a VEV equal to a length parameter R. Because of SO(4) invariance it is possible to rotate the scalar field that gets a VEV to have only the component $X^{4(8)}$. In order to make the notation suitable for the more general case, at this point we re-label the 3-algebra direction "4" as "ϕ". Thus the four indices split into $a \in \{1, 2, 3\}$ and ϕ. The direction ϕ singled out in this manner will shortly be interpreted as the zero-mode.

Because scalar fields have canonical dimension $\frac{1}{2}$ while R has dimension -1, our proposal amounts to saying that:

$$\langle X^{\phi(8)} \rangle = \frac{R}{\ell_p^{3/2}} \, . \tag{3.1}$$

When compactifying M-theory on a circle of radius R to type IIA string theory, the r.h.s. of the above equation turns out to equal $\sqrt{\frac{g_s}{\ell_s}} \equiv g_{\rm YM}$ where g_s, ℓ_s are the string coupling and string length and $g_{\rm YM}$ is the dimensional coupling on D2-branes.

Let us now examine the theory with this VEV. To begin with, note that a VEV $\langle X^{\phi(8)} \rangle$ preserves supersymmetry as long as no other field has a VEV. To see this, consider the fermion variation in eq. (2.8). The first term on the r.h.s. is zero because the scalar VEV is constant while the gauge field VEV is of course zero. The second term vanishes because $X^{\phi(8)}$ can occur at most once in it, while the other two scalar fields have a vanishing VEV. Therefore the theory expanded about this scalar VEV will have maximal supersymmetry. It also depends on a dimensional coupling constant $g_{\rm YM}$ of canonical dimension $\frac{1}{2}$ as expected, and in agreement with the fact that this theory is weakly coupled in the UV and strongly coupled in the IR.

Now let us examine the various terms in the Lagrangean and show how they reproduce the SYM theory. To start with, consider the sextic potential. Introduce the labels $a, b, c \in \{1, 2, 3\}$ as well as $i, j, k \in \{1, 2, \ldots, 7\}$. Then the potential is:

$$\begin{aligned} V(X) &= \frac{1}{12} \sum_{I,J,K=1}^{8} \left(\varepsilon_{ABCD} \varepsilon_{EFG}{}^{D} X^{A(I)} X^{B(J)} X^{C(K)} X^{E(I)} X^{F(J)} X^{G(K)} \right) \\ &= \frac{1}{2} \sum_{i<j}^{7} \left(\varepsilon_{ABCD} \varepsilon_{EFG}{}^{D} X^{A(i)} X^{B(j)} X^{C(8)} X^{E(i)} X^{F(j)} X^{G(8)} \right) \\ &\quad + \frac{1}{2} \sum_{i<j<k}^{7} \left(\varepsilon_{ABCD} \varepsilon_{EFG}{}^{D} X^{A(i)} X^{B(j)} X^{C(k)} X^{E(i)} X^{F(j)} X^{G(k)} \right) \\ &= \frac{1}{2}\, g_{\rm YM}^2 \sum_{i<j}^{7} \left(\varepsilon_{ab\phi d} \varepsilon_{ef\phi}{}^{d} X^{a(i)} X^{b(j)} X^{e(i)} X^{f(j)} \right) + g_{\rm YM} \mathcal{O}\left(X^5\right) + \mathcal{O}\left(X^6\right) \ . \end{aligned} \tag{3.2}$$

In the last line we have inserted the VEV $\langle X^{\phi(8)} \rangle = g_{\rm YM}$, which leads to a term quartic in the remaining X's. Note that in this term, only $X^{a(i)}$ appear where $a \in \{1, 2, 3\}$ and $i \in \{1, 2, \ldots, 7\}$. The terms of order $g_{\rm YM} \mathcal{O}(X^5)$ and $\mathcal{O}(X^6)$ have not been written explicitly because they will decouple at strong coupling.

Using $\varepsilon_{abd\phi} \equiv \varepsilon_{abd}$ where the latter is the 3-index totally antisymmetric symbol and structure constant of an SU(2) Lie algebra, we see that the quartic term becomes:

$$\frac{1}{2}\, g_{\rm YM}^2 \sum_{i<j=1}^{7} \left(\varepsilon_{abc} \varepsilon_{ef}{}^{c} X^{a(i)} X^{b(j)} X^{e(i)} X^{f(j)} \right) \ , \tag{3.3}$$

which is precisely the quartic scalar interaction of maximally supersymmetric SU(2) SYM in $2+1$ dimensions.

Following the same procedure, it is easy to check that the 2-fermion, 2-scalar coupling reduces to the Yukawa coupling of $2+1$ dimensional SYM, plus terms with two fermions and two scalars:

$$\frac{i}{4} \varepsilon_{ABCD} \overline{\Psi}^{B} \Gamma^{IJ} X^{C(I)} X^{D(J)} \Psi^{A} = \frac{i}{2} g_{\rm YM}\, \varepsilon_{abc} \overline{\Psi}^{b} \Gamma^{i} X^{c(i)} \Psi^{a} + \mathcal{O}\left(X^2 \Psi^2\right) \ . \tag{3.4}$$

We see that the only scalars and fermions appearing in the first term (which will be the leading term in the strong coupling limit) are Ψ^a and $X^{a(i)}$.

Since kinetic terms are unaffected by a scalar VEV, it only remains to understand the gauge field terms including couplings of gauge fields through covariant derivatives. On the face of it this should be the major stumbling block, for the gauge field in the 3-algebra theory only has Chern-Simons couplings while the D2-brane Yang-Mills theory requires a dynamical gauge field.[3] We will make no additional assumptions to account for the dynamical gauge field, but simply work out the full content of the theory in the presence of the VEV of the scalar field $X^{\phi(8)}$. We will find that the Higgs mechanism, and the original Chern-Simons coupling, miraculously conspire to provide the desired dynamical gauge field with all the right properties.

In view of our split of indices $A, B \in \{1,2,3,4\}$ into $a, b \in \{1,2,3\}$ and $\phi = 4$, it is natural to break up the gauge field A_μ^{AB} into two parts:

$$A_\mu{}^{a\phi} \equiv A_\mu{}^a \tag{3.5}$$

$$\frac{1}{2}\varepsilon^a{}_{bc}A_\mu{}^{bc} \equiv B_\mu{}^a \,. \tag{3.6}$$

Each of these is a triplet of vector fields. We can now re-write the two terms in the Chern-Simons action as follows:

$$\begin{aligned}\frac{1}{2}\,\varepsilon^{\mu\nu\lambda}\varepsilon_{ABCD}A_\mu{}^{AB}\partial_\nu A_\lambda{}^{CD} &= 2\,\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}A_\mu{}^{ab}\partial_\nu A_\lambda{}^c = 4\,\varepsilon^{\mu\nu\lambda}\,B_\mu{}^a\partial_\nu A_{\lambda\, a}\\ \frac{1}{3}\,\varepsilon^{\mu\nu\lambda}\,\varepsilon_{AEF}{}^G\,\varepsilon_{BCDG}\,A_\mu{}^{AB}A_\nu{}^{CD}A_\lambda{}^{EF} &= -4\,\varepsilon^{\mu\nu\lambda}\,\varepsilon_{abc}B_\mu{}^aA_\nu{}^bA_\lambda{}^c - \frac{4}{3}\,\varepsilon^{\mu\nu\lambda}\,\varepsilon_{abc}B_\mu{}^aB_\nu{}^bB_\lambda{}^c\,.\end{aligned} \tag{3.7}$$

We also need to consider the couplings arising from the covariant derivative on $X^{A(I)}$. We have:

$$\begin{aligned}D_\mu X^{a(I)} &= \partial_\mu X^{a(I)} + \varepsilon^a{}_{BCD}A_\mu{}^{CD}X^{B(I)}\\ &= \partial_\mu X^{a(I)} + 2\,\varepsilon^a{}_{bc}A_\mu{}^cX^{b(I)} + 2\,B_\mu{}^aX^{\phi(I)}\end{aligned} \tag{3.8}$$

and:

$$D_\mu X^{\phi(I)} = \partial_\mu X^{\phi(I)} - 2B_{\mu a}X^{a(I)}\,. \tag{3.9}$$

Inserting these in the Lagrangean (but ignoring fermions) and using the VEV $\langle X^{\phi(8)}\rangle = g_{\rm YM}$, we find the following terms involving $B_\mu{}^a$:

$$\begin{aligned}\mathcal{L}_{\rm kinetic} = &-2g_{\rm YM}^2B_\mu{}^aB_a^\mu - 2B_\mu{}^aX^{\phi(I)}D'^\mu X_a^{(I)} - 2g_{\rm YM}B_\mu{}^aD'^\mu X_a^{(8)}\\ &-2B_{\mu a}X^{a(I)}B_b^\mu X^{b(I)} - 2B_\mu^aB_a^\mu X^{\phi(I)}X_{\phi(I)} + 2B_a^\mu X^{a(I)}\partial_\mu X^{\phi(I)} + \ldots\,,\end{aligned} \tag{3.10}$$

where we have defined a new covariant derivative which depends only on A_μ^a :

$$D'_\mu X^{a(I)} = \partial_\mu X^{a(I)} - 2\varepsilon^a{}_{bc}A_\mu^bX^{c(I)}\,. \tag{3.11}$$

[3] Some kind of non-Abelian duality like $D_\mu X^{a(8)} \sim \varepsilon_\mu{}^{\nu\lambda}F^a_{\nu\lambda}$ has been proposed in the past [8] but so far this has not been possible to implement precisely.

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Notice that the first term is a mass for ${B_\mu}^a$, as one would expect from the Higgs mechanism.

Similarly, the terms involving B_μ^a which come from the gauge field self-couplings are:

$$\mathcal{L}_{\rm CS} = 2\,\varepsilon^{\mu\nu\lambda}\,{B_\mu}^a F'_{\nu\lambda a} - \frac{4}{3}\,\varepsilon^{\mu\nu\lambda}\,\varepsilon_{abc}{B_\mu}^a{B_\nu}^b{B_\lambda}^c + \dots\,, \tag{3.12}$$

where we have also defined:

$$F'^{a}_{\nu\lambda} = \partial_\nu A^a_\lambda - \partial_\lambda A^a_\nu - 2\varepsilon^a{}_{bc}A^b_\nu A^c_\lambda\,. \tag{3.13}$$

Notice that by virtue of its Chern-Simons nature, ${B_\mu}^a$ is an auxiliary field appearing without derivatives. It can therefore be eliminated via its equation of motion. We can extract the leading part of such solution by temporarily neglecting the quadratic term in B^a_μ coming from the cubic self-interaction as well as terms coming from higher interactions with scalars. Later we will show that these would have led to higher-order contributions which are suppressed in the strong coupling limit. We therefore consider the set of couplings:

$$\mathcal{L} = -2g^2_{\rm YM}{B_\mu}^a B^\mu_a - 2g_{\rm YM}{B_\mu}^a D'^\mu X^{(8)}_a + 2\,\varepsilon^{\mu\nu\lambda}\,{B_\mu}^a F'_{\nu\lambda a} + \text{higher order} \tag{3.14}$$

and find that:

$${B_\mu}^a = \frac{1}{2g^2_{\rm YM}}{\varepsilon_\mu}^{\nu\lambda}\,F'^a_{\nu\lambda} - \frac{1}{2g_{\rm YM}}D'_\mu X^{a(8)}\,. \tag{3.15}$$

Thus one of our gauge fields, ${B_\mu}^a$, has been set equal to the field strength of the other gauge field ${A_\mu}^a$ (plus other terms). Together with the fact that ${B_\mu}^a$ has a mass term, we now see that eliminating ${B_\mu}^a$ will provide a standard Yang-Mills kinetic term for ${A_\mu}^a$! This is the desired miracle that promotes the Chern-Simons gauge field ${A_\mu}^a$ into a dynamical gauge field.

Continuing with the computation, the sum of the Chern-Simons gauge field action and the scalar covariant kinetic terms becomes (up to a total derivative):

$$-\frac{1}{g^2_{\rm YM}}F'^a_{\mu\nu}F'^{\mu\nu}_a - \frac{1}{2}\partial_\mu X^{\phi(I)}\partial^\mu X^{(I)}_\phi - \frac{1}{2}D_\mu X^{a(i)}D^\mu X^{(i)}_a + \mathcal{O}(BX\partial X) + \mathcal{O}(B^2X^2) + \mathcal{O}(B^3)\,. \tag{3.16}$$

A re-definition:

$$A \to \frac{1}{2}A\,, \tag{3.17}$$

leads to:

$$D'_\mu X^{a(I)} \to D_\mu X^{a(I)} \equiv \partial_\mu X^{a(I)} - \varepsilon^a{}_{bc}A^b_\mu X^{c(I)} \tag{3.18}$$

and:

$$F'^a_{\mu\nu} \to \frac{1}{2}F^a_{\mu\nu} \equiv \frac{1}{2}\left(\partial_\mu {A_\nu}^a - \partial_\nu {A_\mu}^a - \varepsilon^a{}_{bc}{A_\mu}^b{A_\nu}^c\right)\,. \tag{3.19}$$

Thus eq. (3.16) finally becomes:

$$\begin{aligned}
&-\frac{1}{4g^2_{\rm YM}}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2}\partial_\mu X^{\phi(I)}\partial^\mu X^{(I)}_\phi \\
&\qquad -\frac{1}{2}D_\mu X^{a(i)}D^\mu X^{(i)}_a + \frac{1}{g_{\rm YM}}\mathcal{O}\left(X\partial X\left(F/g_{\rm YM} + DX\right)\right) \\
&\qquad\qquad + \frac{1}{g^2_{\rm YM}}\mathcal{O}\left(X^2\left(F/g_{\rm YM} + DX\right)^2\right) + \frac{1}{g^3_{\rm YM}}\mathcal{O}\left(\left(F/g_{\rm YM} + DX\right)^3\right)\,.
\end{aligned} \tag{3.20}$$

The terms in B^a_μ that we had neglected will lead to higher interactions with increasingly higher powers of $(F/g_{\rm YM} + DX)$ in the numerator and $g_{\rm YM}$ in the denominator.

For the fermions, we easily find that:

$$\frac{i}{2}\bar{\Psi}^A\Gamma^\mu D_\mu\Psi_A \to \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu\Psi_a + \frac{i}{2}\bar{\Psi}^\phi\Gamma^\mu\partial_\mu\Psi_\phi + \text{higher order}\ , \tag{3.21}$$

where D_μ on the l.h.s. is the 3-algebra covariant derivative while D_μ on the right is the Yang-Mills covariant derivative.

The theory we have obtained now has conventional SU(2) Yang-Mills couplings supplemented with some decoupled fields as well as a variety of higher-order terms.[4] The action can be written in the form:

$$\mathcal{L} = \mathcal{L}_{\rm decoupled} + \mathcal{L}_{\rm coupled} \tag{3.22}$$

where

$$\mathcal{L}_{\rm decoupled} = -\frac{1}{2}\partial_\mu X^{\phi(I)}\partial^\mu X^{(I)}_\phi + \frac{i}{2}\bar{\Psi}^\phi\Gamma^\mu\partial_\mu\Psi_\phi\ . \tag{3.23}$$

For the interacting part, we re-scale the fields as $(X,\Psi) \to (X/g_{\rm YM}, \Psi/g_{\rm YM})$, to find the action:

$$\mathcal{L}_{\rm coupled} = \frac{1}{g^2_{\rm YM}}\mathcal{L}_0 + \frac{1}{g^3_{\rm YM}}\mathcal{L}_1 + \mathcal{O}\left(\frac{1}{g^4_{\rm YM}}\right) \tag{3.24}$$

where $\mathcal{L}_0$ is the action of maximally supersymmetric $2+1$ dimensional Yang-Mills theory:

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{4}F_{\mu\nu\,a}F^{\mu\nu\,a} - \frac{1}{2}D_\mu X^{a(i)}D^\mu X_a{}^{(i)} + \frac{1}{4}\left(\varepsilon_{abc}X^{a(i)}X^{b(j)}\right)\left(\varepsilon_{de}{}^c X^{d(i)}X^{e(j)}\right)\\ &+\frac{i}{2}\bar{\Psi}^a\,\not{D}\Psi_a + \frac{i}{2}\varepsilon_{abc}\bar{\Psi}^a\Gamma^i X^{b(i)}\Psi^c\ ,\end{aligned} \tag{3.25}$$

with the field strength and covariant derivative defined as:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon^a{}_{bc}A^b_\mu A^c_\nu \quad \text{and} \quad D^{ab}_\mu = \partial_\mu\delta^{ab} + \varepsilon^{ab}{}_c A^c_\mu\ . \tag{3.26}$$

Note that in the above, $\mathcal{L}_0, \mathcal{L}_1, \ldots$ are all completely independent of $g_{\rm YM}$.

Since the M2-brane is supposed to describe the strongly coupled Yang-Mills theory, we expect it will match on to SYM in the IR limit $g_{\rm YM} \to \infty$. In this limit, we see from eq. (3.24) that the interacting part of the surviving theory is precisely the SU(2) SYM theory on two D2-branes. Note that $B_\mu{}^a$ has disappeared from the theory while $A_\mu{}^a$ no longer has a Chern-Simons coupling but rather a full-fledged SU(2) Yang-Mills action. The fields that survive in the D2-brane action have precisely the right covariant-derivative couplings to the newly-dynamical gauge field. The right quartic and Yukawa couplings have already been obtained at the beginning of this section. Finally, the terms corresponding to the modes $X^{a(8)}$ have disappeared; they have played the role of the Goldstone bosons that gave a mass to B^a_μ and at the end, have transmuted via the Higgs mechanism and the Chern-Simons coupling into the single physical polarisation of $A_\mu{}^a$.

Our final theory also contains 8 non-interacting scalars $X^{\phi(I)}$. Of these, $X^{\phi(i)}, i = 1, 2, \ldots, 7$ correspond to the centre-of-mass modes for the D2 world-volume theory. The

[4] The original 3-algebra still makes its presence in the higher-order terms.

last scalar $X^{\phi(8)}$, the one which originally developed a VEV, can now be dualised via an *Abelian* duality to yield an extra U(1) gauge field. The free Abelian multiplet is completed by Ψ^{ϕ}. The whole multiplet comes from a direction that was *not central* in the original 3-algebra.

One might be alarmed at the fact that the original gauge symmetry $SO(4) \simeq SU(2) \times SU(2)$ appears to have been Higgsed to $SU(2) \times U(1)$ by a VEV of a field in the 4 of SO(4). That is not quite the case. The Higgs mechanism breaks SO(4) to $SO(3) \simeq SU(2)$ as it should, but several free scalars are left over, and the U(1) gauge field is obtained by dualising one of them.

4. M2 to D2: general case

In this section we extend our proposal to more general 3-algebras. We will be hampered by the scant knowledge of 3-algebras but will find that the general case proceeds in much the same way as the $\mathcal{A}_4$ case that we just examined, though there are also some differences.

We start by observing a key feature of the 'fundamental identity' eq. (2.3).[5] By fixing two of the indices to take a specific value, say $E = A = \phi$, and defining 3-index structure constants via $f^{abc} \equiv f^{abc\phi}$, one recovers the Jacobi identity for the usual Lie algebras:

$$f^{dfg} f^{bc}{}_{g} + f^{bfg} f^{cd}{}_{g} + f^{cfg} f^{db}{}_{g} = 0 \, . \tag{4.1}$$

The indices $\{a, b, \ldots\} = 1, \ldots, \dim_{\mathcal{A}} - 1$ run over the dimension of the Lie algebra. We will call this $\mathcal{Q}$, *i.e.* $\dim_{\mathcal{A}} - 1 = \dim_{\mathcal{Q}}$.

In view of our preceding observations, we would like to interpret this fact as saying that after assigning a VEV to one scalar, the remaining directions describe $SU(N)$ degrees of freedom coupled via an SYM theory. This provides some constraints on the 3-algebra, namely all the structure constants $f^{abc\phi}$ are determined. However, as we now see, there are more 3-algebra structure constants to be determined.

Recall that for the $\mathcal{A}_4$ 3-algebra, the structure constants ε^{ABCD} reduced to $\varepsilon^{abc\phi}$, with $a, b, c \in \{1, 2, 3\}$ and obviously there were no components ε^{abcd} left over. This is less obvious in the general case, for which we can allow f^{ABCD} to split into both $f^{abc\phi}$ as well as f^{abcd}. Indeed, if the algebra is not $\mathcal{A}_4$ then f^{abcd} cannot all be zero as we now show. For this, assume the contrary, namely that $f^{abcd} = 0$ for all $a, b, c, d \in \{1, 2, \ldots, \mathcal{A} - 1\}$. In that case choosing A, B, C, D, E, F to be a, b, c, d, e, f in eq. (2.3), we find that the summation index g can only be equal to ϕ. As a result we have the identity:

$$f^{aef} f^{bcd} - f^{bef} f^{acd} + f^{cef} f^{abd} - f^{def} f^{abc} = 0 \, . \tag{4.2}$$

This identity, involving no summation over common indices, certainly does not hold for the structure constants of general Lie algebras. However it does hold for $f^{abc} = \varepsilon^{abc}$ just because the number of possible indices is so small. This shows that in general the assumption $f^{abcd} = 0$ is incompatible with the fundamental identity. Hence, in the general case we will have 3-algebra interactions even among the $X^{a(I)}$s.

[5] We are grateful to Neil Lambert for emphasising this to us.

To recover a D2-brane gauge theory with gauge group SU(N), we assume there exists a 3-algebra with structure constants f^{ABCD}, $A, B, C, D \in \{1, 2, \dots, N^2\}$. We next pick some direction ϕ and identify the structure constants $f^{abc\phi}$ with the f^{abc} of SU(N), where $a, b, c \in \{1, 2, \dots, N^2 - 1\}$. We can now, as before, break up the scalar fields $X^{A(I)}$ and the fermions Ψ^A into the sets $X^{a(I)}, X^{\phi(I)}$ and Ψ^a, Ψ^ϕ. The first step of our reduction procedure is then to postulate that:

$$\langle X^{\phi(8)} \rangle = g_{\rm YM} \,. \tag{4.3}$$

Expanding around this VEV, the sixth order interactions descend to quartic plus higher-order terms:

$$\frac{1}{2} g_{\rm YM}^2 \sum_{i<j=1}^{7} \left(f_{abc} f_{ef}{}^{c} X^{a(i)} X^{b(j)} X^{e(i)} X^{f(j)} \right) + \dots \,, \tag{4.4}$$

just as in the $\mathcal{A}_4$ case. Reduction of the two-fermion, two-scalar coupling to the Yukawa coupling proceeds in the same manner.

Thus the only new feature arises with the gauge fields A_μ^{AB}. We can again split them into the two sets:

$$A_\mu{}^{a\phi} \equiv A_\mu{}^a, \qquad A_\mu{}^{bc} \,. \tag{4.5}$$

Now we naively no longer have equal numbers of components in the two sets. The single-index field $A_\mu{}^a$ has $N^2 - 1$ components. The other field $A_\mu{}^{bc}$ has instead $\frac{(N^2-1)(N^2-2)}{2}$ components, which equals $N^2 - 1$ only for $N^2 = 4$ which is the SU(2) case. In general it has many more components than $A_\mu{}^a$. This appears to contradict the idea of making one part of the gauge field massive via the Higgs mechanism and then, by eliminating that field, rendering the other one dynamical.

However, we are saved by a property of the theory referred to earlier. When we consider the covariant derivatives, we see that the only combinations of gauge fields that appear are $A_\mu{}^a$ and $B_\mu{}^a = \frac{1}{2} f^a{}_{bc} A_\mu{}^{bc}$:

$$D_\mu X^{a(I)} = \partial_\mu X^{a(I)} + f^a{}_{BCD} A_\mu^{CD} X^{B(I)} = \partial_\mu X^{a(I)} + 2 f^a{}_{bc} A_\mu{}^c X^{b(I)} + 2 B_\mu{}^a X^{\phi(I)} \,. \tag{4.6}$$

This is a manifestation of the fact [7] that the theory depends only on $\tilde{A}$ rather than A.

Similarly when we examine the Chern-Simons couplings, we find that the combination of A_μ^{ab} that couples to A_μ^a is precisely:

$$B_\mu{}^a \partial_\nu A_{\lambda\ a} \tag{4.7}$$

plus cubic terms of the form $B \wedge A \wedge A$. Therefore our previous procedure goes through essentially unchanged. The Higgs mechanism causes $X^{a(8)}$ to disappear from the spectrum by giving a mass to $B_\mu{}^a$, and this field is then eliminated by setting it equal to the field strength $F_{\mu\nu}^a$, leading to the promotion of $A_\mu{}^a$ to a dynamical gauge field. A set of free fields $X^{\phi(I)}$ and Ψ^ϕ are left over to generate the decoupled U(1) multiplet. The resulting theory therefore has an SU(N) Yang-Mills part that survives at strong coupling, and an Abelian part.

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The above procedure is less explicit only to the extent that a construction of the 3-algebra structure constants is not known. However it suggests a way to proceed. Given that $f^{abc\phi}$ are completely known, we consider them as "input" for the set of linear equations obtained by putting *one* free index in the fundamental identity equal to ϕ:

$$f^{aef}{}_g\, f^{bc\phi g} - f^{bef}{}_g\, f^{ac\phi g} + f^{cef}{}_g\, f^{ab\phi g} - f^{\phi ef}{}_g\, f^{abcg} = 0 \tag{4.8}$$

which can be re-written as:

$$f^{bcg}\, f^{aef}{}_g + f^{cag}\, f^{bef}{}_g + f^{abg}\, f^{cef}{}_g = f^{efg}\, f^{abc}{}_g\,. \tag{4.9}$$

Treating the 3-index f^{abc} as input, this is a set of linear equations for the unknown quantities f^{abcd}. Solutions to this system of equations should be easier to classify, because of linearity, than solutions of the full fundamental identity. In this sense, the reduction to SU(N) structure constants when one index is set equal to ϕ is like a boundary condition. Finally, one has to ensure that the resulting structure constants satisfy the full fundamental identity, which for the reduced f^{abcd} becomes:

$$\begin{aligned} &f^{aef}{}_g\, f^{bcdg} - f^{bef}{}_g\, f^{acdg} + f^{cef}{}_g\, f^{abdg} - f^{def}{}_g\, f^{abcg} \\ &\qquad = -(f^{aef}\, f^{bcd} - f^{bef}\, f^{acd} + f^{cef}\, f^{abd} - f^{def}\, f^{abc}) \end{aligned} \tag{4.10}$$

We hope to carry out this analysis in the future.

5. Discussion

We have shown that when one component $X^\phi(8)$ of the scalar fields develops a VEV proportional to R, the ensuing Higgs mechanism produces a strongly coupled SYM theory on a *pair* of D2-branes, along with a decoupled theory. The emergence of SYM, complete in all details, from the 3-algebra affirms the relationship of 3-algebra theories to string theory and thereby M-theory. Every interaction of the 3-algebra theory is tested, including its most unusual feature of a Chern-Simons field with a gauge group under which the physical fields apparently transform as fundamental rather than adjoint fields. After Higgsing, part of the Chern-Simons gauge field has become dynamical, and the physical fields are adjoints of this dynamical gauge field.

Let us discuss the physical interpretation of our results.[6] The emergence of D2-brane theories may suggest we are dealing with a compactification of M-theory. Upon compactifying M-theory on a circle of radius R, there will be a periodic array of M2-branes in the x^8 direction. When dealing with D-branes in string theory, one can derive the dynamics explicitly following ref. [13]. An infinite periodic array of the D-brane system along the chosen direction causes the finite matrices on the world-volume to be extended to $\infty \times \infty$ matrices that incorporate the degrees of freedom of strings connecting branes at different places in the periodic array. The result is then an R-dependent action. Next, one quotients

[6]We are grateful to Shiraz Minwalla, Ashoke Sen and David Tong for their helpful comments on a first version of this manuscript.

both the space and the world-volume theory by a translation, which compactifies the direction and turns these strings into winding strings. At the end of this process one finds a set of modes that assemble into the world-volume of a brane of one higher dimension, complete with the extra component of the gauge field. This is the statement of T-duality for multiple D-branes.

For periodic M2-branes, these will all be linked by an infinite dimensional 3-algebra (though one does not expect membrane winding modes when there is a single compact direction). Speculations about the structure of the infinite 3-algebra exist in the literature (see the comments in ref. [11], following earlier ideas of refs. [9, 14, 15] and it seems likely that it will be simpler than a generic finite-dimensional 3-algebra. In the absence of a precise result on this, one interpretation of our result could be that the net effect of compactification is captured by the scalar VEV. In this interpretation the parameter R describing the VEV would be identified with the compactification radius. Then the resulting theory should describe D2-branes. The fact that we get $\mathrm{SU}(2) \times \mathrm{U}(1)$ for the $\mathcal{A}_4$ 3-algebra would indicate that on compactification it describes *two* D2-branes *including their centre-of-mass degree of freedom.* This in turn would mean that before compactification it described two M2-branes including their centre-of-mass mode, which lies within the 3-algebra and is *not central* (in the sense that it does not satisfy $[T^\phi, T^I, T^J] = 0$ for all I, J). This picture avoids the need to postulate triviality of the IR fixed point for two D2-branes. Moreover it generalises in a straightforward way to higher-rank 3-algebras (assuming they exist) and leads to an $\mathrm{SU}(N) \times \mathrm{U}(1)$ theory, which would describe N D2-branes including their centre-of-mass mode.

An objection to this approach is that compactification of a circle is a change of the background and should lead to a *different* world-volume theory instead of the same theory with a different VEV. However the equations we find are very suggestive that the VEV R is related to a compactification radius and we are therefore led to suspect that our Higgsed theory in some way captures the dynamics that would result upon compactification.

One might worry that the non-decoupling of the centre-of-mass from the other degrees of freedom violates physical expectations following from translation invariance. However, while the zero mode is coupled in the 3-algebra, it is not clear that this causes it to couple to *physical, gauge-invariant* degrees of freedom. As an example, in the $\mathcal{A}_4$ 3-algebra, the scalar $X^{\phi(8)}$ that develops a VEV breaks $\mathrm{SU}(2) \times \mathrm{SU}(2)$ to a diagonal $\mathrm{SU}(2)$ under which it is neutral. Hence, as we have seen, after compactification the zero mode does decouple from the remaining modes. But even without compactification of a transverse direction, one would expect that the physical degrees of freedom of the two M2-branes were contained within this diagonal $\mathrm{SU}(2)$, and therefore decoupled — but not manifestly so — from the putative centre-of-mass direction. For the general case, the 3-algebra has an $\mathrm{SO}(N^2)$ structure in which sits an $\mathrm{SU}(N) \times \mathrm{SU}(N)$, further broken to diagonal $\mathrm{SU}(N)$ when there is a scalar VEV. The physical degrees of freedom of N M2-branes, expected to be $\mathcal{O}(N^{3/2})$ in number, should again sit inside this diagonal $\mathrm{SU}(N)$. The overall picture is that the 3-algebra contains vastly *more* "gauge" degrees of freedom than physical ones, so it is quite possible for any chosen zero-mode direction to decouple from all other modes within the physical subspace. If the above picture is correct then decoupling will be visible only quantum-mechanically in the correct gauge-fixed path integral, in sharp contrast to D-branes where it is already manifest in the classical action.[7]

An alternative interpretation of our results is that giving a VEV to a scalar field takes us onto a Coulomb branch where one M2-brane has moved far away from the others, in a theory with no compactification involved. In this case emergence of an $SU(2) \times U(1)$ from the 3-algebra $\mathcal{A}_4$ would be interpreted as describing a pair of strongly coupled M2-branes and a decoupled M2-brane far away (as $R \to \infty$). With this interpretation the 3-algebra $\mathcal{A}_4$ describes *three* M2-branes, as originally envisioned in ref. [7]. However in contrast to ref. [7] we are no longer forced to assume that the IR fixed point on 2 M2-branes is trivial. Instead, it merely has no 3-algebra description. The situation for higher 3-algebras also has puzzling features. In our limit, the Higgsed system is a strongly coupled $SU(N)$ dynamics plus a decoupled $U(1)$ theory and this would be interpreted as the theory on N M2-branes plus a decoupled brane. But it is not clear why the dynamics on the N M2-branes is visible as a Yang-Mills, rather than 3-algebra, theory. And the biggest puzzle is why one needs to decouple a single M2-brane by moving it away, in order to see the $SU(N)$ dynamics on the others. We expect further research will clarify the interpretation of our result and bring about a clearer understanding of the overall picture.

An interesting application of our methods would be to the M2-brane theory when *two* transverse directions are compactified. In this case there will be membrane winding modes somehow linking the original membranes. This may provide a new test, as well as a better understanding, of the 3-algebra structure. This situation is more closely analogous to the one considered in string theory where we have periodic arrays of D-branes connected with winding strings [13]. Another application would be to use the M2 action and our D2 reduction to directly relate the Basu-Harvey solution for the M2 $\perp$ M5 intersection [9] to the D1 $\perp$ D3 intersection as viewed by the D1 world-volume theory [16]. This would require compactifying in one of the directions parallel to the M2's, itself an interesting issue to contend with.

There have been previous attempts to carry out the reduction from multiple M2 to D2 branes and Yang-Mills theory [8, 12]. These papers contain some hints of the fact that some 'special' 3-algebra direction plays a role in obtaining a Yang-Mills theory. Though the setting for our work is a conservative one where loop algebras and non-Abelian duality are not invoked, it would be reasonable to think of our result as a concrete realisation of some of the nice ideas in the above works.

Classifying 3-algebras, or at least finding a large class of them, seems a tractable and worthwhile problem. The connection to $SU(N)$ Lie algebras should be a useful guide. As noted above, the general 3-algebra seems to be governed by a structure like $SO(N^2)$ with $SU(N) \times SU(N)$ sitting inside it. For $N = 4$ this fits together neatly, as locally $SO(4) \simeq SU(2) \times SU(2)$. However, for higher N it has been observed in ref. [11] that there are no invariant 4th rank tensors and therefore f^{ABCD} must have a nontrivial kernel. Here we have not addressed the question of how to find this kernel, in other words what substructure of $SO(N^2)$ is relevant. This is again tied to the problem of characterising the physical, gauge invariant subspace of the theory.

[7] We are grateful to Shiraz Minwalla for a discussion on this point.

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